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A Remark on Nonlocal Symmetries for the Calogero–Degasperis–Ibragimov–Shabat Equation

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Abstract

We consider the Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation and find the complete set of its nonlocal symmetries depending on the local variables and on the integral of the only local conserved density of the equation in question. The Lie algebra of these symmetries turns out to be a central extension of that of local generalized symmetries.

1 Introduction

The existence of infinite-dimensional Lie algebra of commuting higher order symmetries for a system of PDEs is well known to be one of the most important signs of its integrability, see e.g. [3, 14, 19, 20]. This algebra can be extended, see e.g. [3], to a noncommutative algebra (which is often referred as a *hereditary algebra*, see [3] and references therein) of polynomial-in-time (and possibly nonlocal) symmetries. In (2+1) dimensions these symmetries are polynomials in time t of arbitrarily high degree, while in (1+1) dimensions one usually can construct only the symmetries which are at most linear in time [3].

For a long time the only known (1+1)-dimensional nonlinear evolution equation possessing symmetries being polynomials in time of arbitrarily high degree was the Burgers equation, see [23] for the complete description of its symmetry algebra. It is natural to ask whether there exist other (1+1)-dimensional evolution equations having the same property. In [18] we have answered this question in affirmative and shown that the Calogero–Degasperis–Ibragimov–Shabat (CDIS) [4, 5, 9, 16, 19, 21] equation also possesses a hereditary algebra of polynomial-in-time symmetries which, exactly as in the case of Burgers equation, is its complete symmetry algebra in the class of *local* higher order symmetries.

The CDIS equation (2.1) has only one nontrivial local conserved density $\rho = u^2$, so the natural next step in analyzing this equation is to consider its symmetries involving a nonlocal variable ω being the integral of this density. Theorem 1 below provides the

complete characterization of symmetries which depend on this variable and on a finite number of local variables. It turns out that, unlike the local case, the Lie algebra of these symmetries possesses a nontrivial one-dimensional center, spanned by (the only) genuinely nonlocal symmetry.

2 Symmetries of the CDIS equation

The Calogero–Degasperis–Ibragimov–Shabat (CDIS) equation has the form [4, 9]

$$u_t = u_3 + 3u^2u_2 + 9uu_1^2 + 3u^4u_1 \equiv F, \tag{2.1}$$

Here $u_j = \partial^j u/\partial x^j$ and $u_0 \equiv u$; see [18] for the further details on notation used.

Let us mention that (2.1) is the only equation among third order (1+1)-dimensional scalar polynomial λ-homogeneous evolution equations with $\lambda = 1/2$ that possesses infinitely many x, t-independent local generalized symmetries [15].

Consider a nonlocal variable ω defined (cf. e.g. [21, 22, 23]) by the relations

$$\partial\omega/\partial x = u^2, \qquad \partial\omega/\partial t = 2uu_2 + 6u^3u_1 + u^6 - u_1^2. \tag{2.2}$$

Note that the CDIS equation is linearized into $v_t = v_3$ upon setting $v = \exp(\omega)u$ [19].

The quantity $\rho = u^2$ is [19, 21] the only nontrivial local conserved density for (2.1), but (2.1) has [21] a Noether operator, i.e., an operator that sends conserved covariants to symmetries, see e.g. [8] for more details on such operators, of the form $\exp(-2\omega)$ and infinitely many nontrivial conserved densities explicitly dependent on ω [21].

We shall call a function $G(x, t, \omega, u, u_1, \ldots, u_k)$ a *symmetry* of CDIS equation, if

$$D_t(G) - F_*(G) = 0, \tag{2.3}$$

where $F_* = \sum\limits_{i=0}^{3} \partial F/\partial u_i D^i$, and $D \equiv D_x = \partial/\partial x + u^2\partial/\partial\omega + \sum\limits_{i=0}^{\infty} u_{i+1}\partial/\partial u_i$ and $D_t = \partial/\partial t + \left(2uu_2 + 6u^3u_1 + u^6 - u_1^2\right)\partial/\partial\omega + \sum\limits_{i=0}^{\infty} D^i(F)\partial/\partial u_i$ are the operators of total x- and t-derivatives. Note that our definition of nonlocal symmetries is a particular case of the usual one, cf. e.g. [3], but in terminology of [22] the solutions of (2.3) are referred as *shadows* of symmetries.

For any function $H = H(x, t, \omega, u, u_1, \ldots, u_q)$ we define its *order* $\operatorname{ord} H$ as a greatest integer m such that $\partial H/\partial u_m \neq 0$, and set

$$H_* = 2\partial H/\partial\omega D^{-1} \circ u + \sum_{i=0}^{\operatorname{ord} H} \partial H/\partial u_i D^i.$$

Here $\circ$ denotes a composition law induced by ‘generalized Leibnitz rule’ (see e.g. [12, 19])

$$D^k \circ f = \sum_{j=0}^{\infty} \frac{k(k-1)\cdots(k-j+1)}{j!} D^j(f) D^{k-j}.$$

A function $H = H(x, t, \omega, u, u_1, \ldots, u_q)$ is called *local* if $\partial H/\partial\omega = 0$.

Let $S^{(k)}_{\mathrm{CDIS}}$ be the set of symmetries of the form $G(x,t,\omega,u,u_1,\ldots,u_k)$ for the CDIS equation, and let $S_{\mathrm{CDIS}} = \bigcup_{j=0}^{\infty} S^{(j)}_{\mathrm{CDIS}}$, $S_{\mathrm{CDIS},j} = S^{(j)}_{\mathrm{CDIS}}/S^{(j-1)}_{\mathrm{CDIS}}$ for $j \geq 1$, and $S_{\mathrm{CDIS},0} = S^{(0)}_{\mathrm{CDIS}}/\Theta_{\mathrm{CDIS}}$, where $\Theta_{\mathrm{CDIS}} = \{G(x,t,\omega)|G \in S_{\mathrm{CDIS}}\}$.

Suppose that $k \equiv \operatorname{ord} G \geq 1$. Then differentiating the left-hand side of (2.3) with respect to u_{k+2} and equating the result to zero yields $D(\partial G/\partial u_k) = 0$. Hence, in analogy with Section 5.1 of [14], for any symmetry $G \in S_{\mathrm{CDIS}}$ of order $k \geq 1$ we have

$$\partial G/\partial u_k = c_k(t), \tag{2.4}$$

where $c_k(t)$ is a function of t.

In particular, any symmetry $G \in S^{(2)}_{\mathrm{CDIS}}$ is of the form $G = c(t)u_2 + g(x,t,\omega,u,u_1)$. Substituting this expression into (2.3), collecting the coefficients at u_4, u_3, u_2, equating to zero these coefficients and the sum of remaining terms on the left-hand side of (2.3), and solving the resulting system of equations for $c(t)$ and g readily shows that $c(t) = 0$ and G is a linear combination (with constant coefficients) of u_1 and $W = \exp(-2\omega)u$.

Below we assume without loss of generality that any symmetry $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, vanishes if the relevant function $c_k(t)$ is identically equal to zero.

Further differentiating (2.3) with respect to u_{k+1} and u_k and then with respect to x shows that, exactly as in [17], for $k \geq 2$ we have $\partial^2 G/\partial u_{k-1}\partial x = 0$, and for $k \geq 3$

$$\partial^2 G/\partial x \partial u_{k-2} = \dot{c}_k(t)/3. \tag{2.5}$$

Taking into account that $G \in S_{\mathrm{CDIS}}$ implies $\tilde{G} = \partial^r G/\partial x^r \in S_{\mathrm{CDIS}}$, and successively using (2.5), we find that for $k \geq 2$ and $r \leq [k/2]-1$ we have $\operatorname{ord}\tilde{G} \leq k-2r$ and

$$\partial \tilde{G}/\partial u_{k-2r} = (1/3)^r d^r c_k(t)/dt^r. \tag{2.6}$$

For $r = [k/2]-1$ we have $\operatorname{ord}\tilde{G} \leq 3$. The straightforward computation shows that all symmetries from $S^{(3)}_{\mathrm{CDIS}}$ are at most linear in t, and thus the function $c_k(t)$ satisfies the equation $d^m c_k(t)/dt^m = 0$ for $m = [k/2]+1$. Hence, $\dim S_{\mathrm{CDIS},k} \leq [k/2]+1$ for $k \geq 1$, i.e., the dimension of the quotient space of symmetries of the form $G = G(x,t,\omega,u,u_1,\ldots,u_k)$ modulo the space of symmetries of the form $G = G(x,t,\omega,u,u_1,\ldots,u_{k-1})$ does not exceed $[k/2]+1$ for $k \geq 1$. From this it is immediate that all odd-order symmetries from S_{CDIS} are exhausted by local ones, as we can exhibit exactly $[k/2]+1$ such symmetries of order k for each odd k [18].

Furthermore, as all symmetries of the form $G(x,t,\omega,u,u_1,u_2)$ are exhausted by u_1 and W, in analogy with Theorem 2 of [17] we can show that all symmetries of the form $G(x,t,\omega,u,u_1,\ldots,u_k)$ of the CDIS equation are polynomial in time t for all $k \in \mathbb{N}$. Indeed, assume this result to be proved for the symmetries of order $k-1$ and let us prove it for symmetries of order k. It readily follows from the above that the function $c_k(t) = \partial G/\partial u_k$ is a polynomial in t of degree not higher than $[k/2]$. Therefore, $\partial^m G/\partial t^m$, where $m = [k/2]+1$, is a symmetry of CDIS of order not higher then $k-1$ and thus is polynomial in t by assumption, whence we readily see that G is polynomial in t as well. The induction on k, starting from $k = 2$, completes the proof.

Now let us turn to the study of time-independent symmetries of the CDIS equation. This equation is well known to have infinitely many x,t-independent local generalized

symmetries, hence it has [10] a formal symmetry of infinite rank of the form $\mathfrak{L} = D + \sum_{j=0}^{\infty} a_j D^{-j}$, where a_j are some x, t-independent local functions.

Taking the directional derivative of (2.3), we find that for any symmetry $G \in S_{\mathrm{CDIS}}$ of order k the quantity G_* is a formal symmetry of rank not lower than $k+1$ for the CDIS equation, and therefore (cf. e.g. [12, 19]), provided $k \geq 1$ and $\partial G/\partial t = 0$, we have

$$G_* = \sum_{j=1}^{k} c_j \mathfrak{L}^j + \mathfrak{B},$$

where c_j are some constants and $\mathfrak{B} = \sum\limits_{j=-\infty}^{0} b_j D^j$, b_j are some t-independent local functions.

From this equation we infer that (cf. [18]) any symmetry $G(x, \omega, u, u_1, \ldots, u_k) \in S_{\mathrm{CDIS}}$, $\operatorname{ord} G \geq 1$, can be written as

$$G = G_0(u, u_1, \ldots, u_k) + Y(x, u, \omega). \tag{2.7}$$

It can be easily seen that $\partial Y/\partial x = \partial G/\partial x$ and $\partial Y/\partial \omega = \partial G/\partial \omega$ are time-independent symmetries of the CDIS equation of order not higher than zero. We readily conclude from the above that $\partial Y/\partial x = c_1 W$ and $\partial Y/\partial \omega = c_2 W$ for some constants c_1, c_2. As $\partial^2 Y/\partial x \partial \omega = \partial^2 Y/\partial \omega \partial x$, we find that $c_1 = 0$, and thus $\partial Y/\partial x = 0$, so any time-independent symmetry G of order $k \geq 1$ for the CDIS equation is x-independent as well, and $G = G_0(u, u_1, \ldots, u_k) + cW$ for some constant c. As we have already shown above, the only symmetry of order zero or less from S_{CDIS} is W. Thus, we conclude that any time-independent symmetry of the form $G(x, \omega, u, u_1, \ldots, u_k)$ is x-independent as well.

Using the symbolic method, it can be shown [15] that the CDIS equation has no even order t, x-independent local generalized symmetries, so its only even order time-independent symmetry (in the class of symmetries of the form $G(x, \omega, u, u_1, \ldots, u_k)$) is W.

Now let us show that the same result holds true for time-dependent symmetries as well. The CDIS equation is invariant under the scaling symmetry $K = 3tF + xu_1 + u/2$. Therefore, if a symmetry Q contains the terms of weight γ (with respect to the weighting induced by K, cf. [2], when the weight of u is $1/2$, the weight of ω is 0, the weight of t is -3, the weight of x is -1, and the weight of $u_j = j+1/2$), there exists a homogeneous symmetry $\tilde{Q}$ of the same weight γ. We shall write this as $\mathrm{wt}(\tilde{Q}) = \gamma$. Note that $[K, \tilde{Q}] = (\gamma - 1/2)\tilde{Q}$.

If $G \in S_{\mathrm{CDIS},k}$, $k \geq 1$, is a polynomial in t of degree m, then its leading coefficient $\partial G/\partial u_k = c_k(t)$ also is a polynomial in t of degree $m' \leq m$, i.e., $c_k(t) = \sum\limits_{j=0}^{m'} t^j c_{k,j}$, where $c_{k,m'} \neq 0$. Consider $\tilde{G} = \partial^{m'} G/\partial t^{m'} \in S_{\mathrm{CDIS}}^{(k)}$. We have $\partial \tilde{G}/\partial u_k = \mathrm{const} \neq 0$, hence $\tilde{G}$ contains the terms of the weight $k+1/2$. Let P be the sum of all terms of weight $k+1/2$ in $\tilde{G}$. Clearly, P is a homogeneous symmetry of weight $k+1/2$ by construction, $\operatorname{ord} P = k$ and $\partial P/\partial u_k$ is a nonzero constant. Next, $\partial P/\partial t \in S_{\mathrm{CDIS}}$ is a homogeneous symmetry of weight $k+7/2$. Obviously, $\operatorname{ord} \partial P/\partial t \leq k-1$. By the above, all symmetries in S_{CDIS} are polynomial in t, and thus for any homogeneous $B \in S_{\mathrm{CDIS}}$, $b \equiv \operatorname{ord} B \geq 1$, we have $\partial B/\partial u_b = t^r c_b$, $c_b = \mathrm{const}$ for some $r \geq 0$. Hence, $\mathrm{wt}(B) = b - 3r + 1/2 \leq b + 1/2$, and for $k \geq 1$ the set S_{CDIS} does not contain homogeneous symmetries B such that $\mathrm{wt}(B) = k + 7/2$ and $\operatorname{ord} B \leq k-1$, so $\partial P/\partial t = 0$.

Thus, we conclude that the existence of a time-independent symmetry of order $k \geq 1$ from $S_{\rm CDIS}$ is a necessary condition for the existence of a polynomial-in-time symmetry $G \in S_{\rm CDIS}$ of the same order k. Moreover, by the above all symmetries from $S_{\rm CDIS}$ are polynomial in t. Hence, the fact that the CDIS equation has no time-independent symmetries $G(x, \omega, u, u_1, \ldots, u_k)$ of even order $k \geq 2$ immediately implies the absence of any *time-dependent* symmetries of even order $k \geq 2$ belonging to $S_{\rm CDIS}$.

Summing up the above results, we infer that the space $S_{\rm CDIS}$ is spanned by the symmetry $W = \exp(-2\omega)u$, and by *local* generalized symmetries of the CDIS equation. The latter were found in [18], and can be described in the following way.

Define the commutator of two functions f and g of $x, t, \omega, u, u_1, \ldots$ as (cf. e.g. [3, 12, 19])

$$[f, g] = g_*(f) - f_*(g).$$

Set $\tau_{m,0} = x^m u_1 + m x^{m-1} u/2$, $m = 0, 1, 2\ldots$, and $\tau_{1,1} = x(u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1) + 3u_2/2 + 5u_1 u^2 + u^5/2$. The latter is the first nontrivial master symmetry for the CDIS equation, see e.g. [15, 16]. The quantities $\tau_{0,0}$, $\tau_{1,0}$, $\tau_{2,0}$, and $\tau_{1,1}$ meet the requirements of Theorem 3.18 from [3], whence

$$[\tau_{m,j}, \tau_{m',j'}] = ((2j'+1)m - (2j+1)m')\tau_{m+m'-1,j+j'}, \tag{2.8}$$

where $\tau_{m,j}$ with $j > 0$ are defined inductively by means of (2.8), i.e., $\tau_{0,j+1} = \frac{1}{2j+1}[\tau_{1,1}, \tau_{0,j}]$, $\tau_{m+1,j} = \frac{1}{2+4j-m}[\tau_{2,0}, \tau_{m,j}]$, see [3].

Note that the idea of constructing new symmetries by the repeated commutation of a master symmetry with a seed symmetry, as well as the notion of master symmetry, were suggested by Fuchssteiner and Fokas [6], see also Fuchssteiner [7].

Thus, the CDIS equation, as well as the Burgers equation, represents a nontrivial example of a (1+1)-dimensional evolution equation possessing a "doubly infinite" hereditary algebra of master symmetries $\tau_{m,n}$.

Using (2.8), it can be shown (cf. [3]) that $\mathrm{ad}_{\tau_{0,j}}^{m+1}(\tau_{m,j'}) = 0$, i.e., $\tau_{m,j}$ are master symmetries of degree m for all equations $u_{t_k} = \tau_{0,k}$, $k = 0, 1, 2, \ldots$. Here $\mathrm{ad}_B(G) \equiv [B, G]$ for any (smooth) functions B and G depending on ω and on a finite number of local variables.

Let $\exp(\mathrm{ad}_B) \equiv \sum\limits_{j=0}^{\infty} \mathrm{ad}_B^j / j!$. As $\mathrm{ad}_{\tau_{0,j}}^{m+1}(\tau_{m,j'}) = 0$, it is easy to see (cf. [3]) that

$$\begin{aligned} G_{m,j}^{(k)}(t_k) &= \exp(-t_k \,\mathrm{ad}_{\tau_{0,k}})\tau_{m,j} \\ &= \sum_{i=0}^{m} \frac{(-t_k)^i}{i!} \mathrm{ad}_{\tau_{0,k}}^i(\tau_{m,j}) = \sum_{i=0}^{m} \frac{((2k+1)t_k)^i m!}{i!(m-i)!} \tau_{m-i,j+ik} \end{aligned}$$

are local time-dependent generalized symmetries for the equation $u_{t_k} = \tau_{0,k}$, and $\mathrm{ord}\, G_{m,j}^{(k)} = 2(j + mk) + 1$. Note that $G_{m,j}^{(k)}$ obey the same commutation relations as $\tau_{m,j}$, that is,

$$[G_{m,j}^{(k)}, G_{m',j'}^{(k)}] = ((2j'+1)m - (2j+1)m')G_{m+m'-1,j+j'}^{(k)}. \tag{2.9}$$

It is straightforward to verify that $\tau_{0,1} = F = u_3 + 3u^2 u_2 + 9uu_1^2 + 3u^4 u_1$ and thus $G_{m,j} = G_{m,j}^{(1)}(t) = \exp(-t \,\mathrm{ad}_F)\tau_{m,j}$ are time-dependent symmetries for the CDIS equation.

It is easy to see that the number of symmetries $G_{m,j}$ of given odd order $k = 2l + 1$ equals $[k/2] + 1 = l + 1$. As $\dim S_{\mathrm{CDIS},k} \leq [k/2] + 1$, these symmetries exhaust the space $S_{\mathrm{CDIS},k}$ for odd k. Moreover [18], any local generalized symmetry of the CDIS equation is a linear combination of the symmetries $G_{m,j}$ for $m = 0, 1, 2, \ldots$ and $j = 0, 1, 2 \ldots$.

Evaluating the commutator of W with $\tau_{m,0} = x^m u_1 + m x^{m-1} u/2$, we readily see that it vanishes, if we assume that the result of application of D^{-1} to a homogeneous polynomial in $x, t, u, \ldots, u_k$ without u_j-independent terms is a polynomial of the same kind. Next, under the same assumption $[W, \tau_{0,1}] = [W, F] = 0$, just because $W \in S_{\mathrm{CDIS}}$. Finally, using the commutation relations (2.8) and the Jacobi identity, we conclude that the commutator of W with all symmetries $G_{m,j}$ vanishes as well. As a result, we have

Theorem 1. *Any symmetry of the CDIS equation of the form $G(x, t, \omega, u, \ldots, u_s)$ is a linear combination of the symmetries $G_{m,j}$ for $m, j = 0, 1, 2, \ldots,$ and of the symmetry $W = \exp(-2\omega)u$, which commutes with all other symmetries.*

This result can be generalized to the symmetries of 'higher CDIS equations' $u_{t_k} = \tau_{0,k}$. Recall [10] that because of existence of infinitely many x, t-independent local generalized symmetries the CDIS equation has a formal symmetry of infinite rank of the form $\mathfrak{L} = D + u^2 + \sum_{j=1}^{\infty} a_j D^{-j}$, where a_j are some x, t-independent local functions. Using Lemma 11 from [20], we can show that $\mathfrak{L}$ is a formal symmetry of rank at least $\operatorname{ord} \tau_{0,k} + 3$ for the equation $u_{t_k} = \tau_{0,k}$, and hence $u^2 = \operatorname{res} \ln \mathfrak{L}$ is a conserved density for all these equations: $D_{t_k}(u^2) = D(\sigma_k)$, where σ_k are some local functions. Note that σ_k can be choosen to be polynomials in u_j with x, t-independent coefficients and zero free term (in particular, we have chosen $\sigma_1 = 2uu_2 + 6u^3 u_1 + u^6 - u_1^2 = \partial\omega/\partial t$ in (2.2) to be exactly of this form).

In order to describe the symmetries of the systems $u_{t_k} = \tau_{0,k}$, let us extend (2.2) by adding the equations

$$\partial\omega/\partial t_k = \sigma_k.$$

Theorem 2. *Any symmetry of the form $G(x, t_k, \omega, u, \ldots, u_s)$ for the equation $u_{t_k} = \tau_{0,k}$, $k \geq 1$, is a linear combination of the symmetries $G^{(k)}_{m,j}(t_k)$ for $m = 0, 1, 2, \ldots$ and $j = 0, 1, 2 \ldots,$ and of $W = \exp(-2\omega)u$, which commutes with all other symmetries.*

As a final remark, let us mention that, as an alternative approach, one could try using the formula $v = \exp(\omega)u$ in order to "transfer" the known results on symmetries and recursion operators of the linear equation $v_t = v_3$ to the CDIS equation. However, the symmetries resulting in this way are in general highly nonlocal. Even the description of nonlocalities that occur in thus constructed symmetries, as well as sorting out local and "weakly nonlocal" (depending only on ω) symmetries, is quite a difficult task, and we intend to analyze this and related problems elsewhere.

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References

[1] Adler V E, Shabat A B, Yamilov R I, Symmetry Approach to the Integrability Problem, *Theor. Math. Phys.* **125**, Nr. 3 (2000), 1603–1661.

[2] Beukers F, Sanders J A and Wang J P, One Symmetry Does Not Imply Integrability, *J. Diff. Eqns* **146** (1998), 251–260.

[3] Błaszak M, Multi-Hamiltonian Dynamical Systems, Springer, Berlin, 1998.

[4] Calogero F and Degasperis A, Reduction Technique for Matrix Nonlinear Evolution Equations Solvable by the Spectral Transform, Preprint 151, Istituto di Fisica G. Marconi Univ. di Roma, 1979, published in *J. Math. Phys.* **22** (1981), 23–31.

[5] Calogero F, The Evolution Partial Differential Equation $u_t = u_{xxx} + 3(u_{xx}u^2 + 3u_x^2 u) + 3u_x u^4$, *J. Math. Phys.* **28** (1987), 538–555.

[6] Fokas A S and Fuchssteiner B, The Hierarchy of the Benjamin–Ono Equation, *Phys. Lett.* **A86** (1981), Nr. 6–7, 341–345.

[7] Fuchssteiner B, Mastersymmetries, Higher Order Time-Dependent Symmetries and Conserved Densities of Nonlinear Evolution Equations, *Progr. Theor. Phys.* **70** (1983), 1508–1522.

[8] Fuchssteiner B and Fokas A S, Symplectic Structures, their Bäcklund Transformations and Hereditary Symmetries, *Phys.* **D4** (1981/82), Nr. 1, 47–66.

[9] Ibragimov N H and Shabat A B, Infinite Lie–Bäcklund Algebras, *Funct. Anal. Appl.* **14** (1981), 313–315.

[10] Ibragimov N H, Transformation Groups Applied to Mathematical Physics, Reidel Publishing Co., Dordrecht, 1985.

[11] Mikhailov A V, Shabat A B and Sokolov V V, The Symmetry Approach to Classification of Integrable Equations, in What is Integrability?, Editor: Zakharov V E, Springer, New York, 1991, 115–184.

[12] Mikhailov A V, Shabat A B and Yamilov R I, The Symmetry Approach to Classification of Nonlinear Equations. Complete Lists of Integrable Systems, *Russ. Math. Surv.* **42** (1987), Nr. 4, 1–63.

[13] Mikhailov A V and Yamilov R I, Towards Classification of (2 + 1)-Dimensional Integrable Equations. Integrability Conditions. I, *J. Phys. A: Math. Gen.* **31** (1998), 6707–6715.

[14] Olver P J, Applications of Lie Groups to Differential Equations, Springer, New York, 1986.

[15] Sanders J A and Wang J P, On the Integrability of Homogeneous Scalar Evolution Equations, *J. Diff. Eqns* **147** (1998), 410–434.

[16] Sanders J A and Wang J P, On Integrability of Evolution Equations and Representation Theory, *Cont. Math.* **285** (2001), 85–99.

[17] Sergyeyev A, On Symmetries of KdV-like Evolution Equations, *Rep. Math. Phys.* **44** (1999), 183–190.

[18] Sergyeyev A and Sanders J A, The Complete Set of Generalized Symmetries for the Calogero–Degasperis–Ibragimov–Shabat Equation, Proc. 4th Int. Conf. "Symmetry in Nonlinear Mathematical Physics", Editors: Nikitin A G, Boyko V M and Popovych R O, in *Proc. Inst. Math. NASU*, Vol. 43, Part 1, Institute of Mathematics of NASU, Kyiv, 2002, 209–214.

[19] Sokolov V V and Shabat A B, Classification of Integrable Evolution Equations, *Sov. Sci. Rev., Sect. C, Math. Phys. Rev.* **4** (1984), 221–280.

[20] Sokolov V V, On the Symmetries of Evolution Equations, *Russ. Math. Surv.* **43** (1988), Nr. 5, 165–204.

[21] Svinolupov S I and Sokolov V V, On Conservation Laws for Equations Having a Non-Trivial Lie–Bäcklund Algebra, in Integrable Systems, Editor: Shabat A B, Bashkirian Branch of Academy of Sciences of USSR, Ufa, 1982, 53–67.

[22] Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Editors: Krasil'shchik I S and Vinogradov A M, Translations of Mathematical Monographs, Vol. 182, American Mathematical Society, Providence, RI, 1999.

[23] Vinogradov A M and Krasil'shchik I S, A Method of Computing Higher Symmetries of Nonlinear Evolution Equations, and Nonlocal Symmetries, *Sov. Math. Dokl.* **22** (1980), 235–239.